Problem Set #4, Solutions

1 (a) $\dot{x} = \frac{p}{m}$, $\dot{p} = -\gamma \frac{p}{m} + \Xi(t)$

$$F(t) = q E_0 \cos\omega t = -\frac{\partial}{\partial x} V(x,t)$$
$$V(x,t) = -q E_0 x \cos\omega t$$

$$\mathcal{H}(x,p,t) = \underbrace{\frac{p^2}{2m}}_{\mathcal{H}_0} - \underbrace{q E_0 x \cos\omega t}_{\epsilon\,\alpha(t)\,A(x)}$$

We can set $\epsilon = q E_0$, $\alpha(t) = \cos\omega t$, $A(x) = x$

Setting $B(x,p) = p$, we have:

$$\langle p \rangle_t = \langle B \rangle_t = \epsilon \int_0^t ds\ \alpha(s)\, R_{AB}(t-s)$$

where $R_{AB}(t) = \beta \langle \dot{A}_0 B_t \rangle = \beta \langle \dot{x}_0 p_t \rangle$

$$= \frac{\beta}{m} \langle p_0 p_t \rangle = \frac{\beta}{m} \frac{m}{\beta} e^{-\gamma|t|/m}$$

$$= e^{-\gamma|t|/m}$$

$$\langle p \rangle_t = q E_0 \int_0^t ds\ \cos(\omega s)\, e^{-\gamma|t-s|/m}$$
$$= q E_0\, \mathrm{Re} \int_0^t ds\ e^{i\omega s} e^{-\gamma(t-s)/m}$$
$$= q E_0\, e^{-\gamma t/m}\, \mathrm{Re} \int_0^t ds\ e^{(i\omega + \frac{\gamma}{m})s}$$

$$= q E_0 \, e^{-\gamma t/m} \, \mathrm{Re} \left. \frac{e^{\left(\frac{\gamma}{m} + i\omega\right)s}}{\frac{\gamma}{m} + i\omega} \right|_0^t$$

$$= q E_0 \, e^{-\gamma t/m} \, \mathrm{Re} \, \frac{e^{\left(\frac{\gamma}{m} + i\omega\right)t} - 1}{\frac{\gamma}{m} + i\omega}$$

$$= q E_0 \, \mathrm{Re} \, \frac{e^{i\omega t} - e^{-rt}}{r + i\omega} \qquad r = \frac{\gamma}{m}$$

$$= \frac{q E_0}{r^2 + \omega^2} \cdot \left(r \cos\omega t + \omega \sin\omega t - r e^{-rt} \right)$$

$$= \frac{q E_0}{\sqrt{r^2 + \omega^2}} \left(\frac{r \cos\omega t + \omega \sin\omega t}{\sqrt{r^2 + \omega^2}} - \frac{r e^{-rt}}{\sqrt{r^2 + \omega^2}} \right)$$

$$\cancel{= \frac{q E_0}{\sqrt{r^2 + \omega^2}} \left(\tan\phi - \frac{r e^{-rt}}{\sqrt{r^2 + \omega^2}} \right)}$$

$$= \frac{q E_0}{\sqrt{\frac{\gamma^2}{m^2} + \omega^2}} \left[\cos(\omega t - \phi) - \frac{\frac{\gamma}{m} e^{-\gamma t/m}}{\sqrt{\frac{\gamma^2}{m^2} + \omega^2}} \right]$$

$$\tan\phi = \frac{m\omega}{\gamma}$$

$$\boxed{\langle p \rangle_t = \psi \cos(\omega t - \phi) - \psi \cos(\phi) \, e^{-\gamma t/m} \qquad \psi = q E_0 \Big/ \sqrt{\frac{\gamma^2}{m^2} + \omega^2}}$$

(b) $t \gg \frac{m}{\gamma} \rightarrow \langle p \rangle_t = \psi \cos(\omega t - \phi)$, in agreement w/ Prob. Set. #3

2 a $$\frac{\partial f}{\partial t} = \beta D \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x} f\right) + D\frac{\partial^2 f}{\partial x^2} = \hat{\mathcal{L}} f \quad , \quad V = \frac{1}{2}kx^2$$

Using correspondence between Fokker-Planck & Schrödinger operators: (see 3/12 notes)

$$-\frac{\partial g}{\partial t} = -D\frac{\partial^2 g}{\partial x^2} + D V_{eff}\, g \equiv \hat{H} g$$

$$V_{eff} = \left(\frac{\beta}{2}\frac{\partial V}{\partial x}\right)^2 - \frac{\beta}{2}\frac{\partial^2 V}{\partial x^2} = \frac{\beta^2 k^2 x^2}{4} - \frac{\beta k}{2}$$

rewrite $\hat{H}$ in familiar form:

$$D \to \frac{\hbar^2}{2m} \quad , U(x) = D V_{eff} \to \frac{\beta^2\hbar^2 k^2 x^2}{8m} - \frac{\beta \hbar^2 k}{4m}$$

$$= \frac{m\omega^2}{2}x^2 - U_0 \quad ; \omega = \frac{\beta\hbar k}{2m} \,, U_0 = \frac{\hbar\omega}{2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2 - U_0$$

This is the quantum harmonic oscillator Hamiltonian, with offset $-U_0 = -\hbar\omega/2$.

Eigenstates & eigenvalues of $\hat{H}$: $n = 0, 1, 2, \cdots$

$$\begin{cases} \psi_n(x) = \dfrac{1}{\sqrt{2^n n!}}\left(\dfrac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}\, H_n\!\left(\sqrt{\dfrac{m\omega}{\hbar}}\, x\right) \\ E_n = \left(n+\frac{1}{2}\right)\hbar\omega - U_0 = n\hbar\omega \end{cases}$$

Hermite polynomial $\left[H_n(z) = (-1)^n e^{z^2}\frac{\partial^n}{\partial z^n}e^{-z^2}\right]$

Translate back to F-P operator:

eigenstates of $\hat{\mathcal{L}}$: $\varphi_n(x) = \sqrt{\pi(x)}\,\psi_n(x)$

eigenstates of $\hat{\mathcal{L}}^+$: $\alpha_n(x) = \frac{1}{\sqrt{\pi(x)}}\varphi_n(x)$

eigenvalues : $\lambda_n = -E_n = -n\hbar\omega$

Rewrite in terms of original variables (β, D, k):

$$\boxed{\begin{aligned} \varphi_n(x) &= \frac{1}{\sqrt{2^n n!}} \sqrt{\frac{\beta k}{2\pi}}\, e^{-\beta k x^2/2}\, H_n\left(\sqrt{\frac{\beta k}{2}}\, x\right) \\ \alpha_n(x) &= \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{\beta k}{2}}\, x\right) \\ \lambda_n &= -n\beta D k \end{aligned}}$$

Note: $\varphi_0(x) = \pi(x) = \sqrt{\frac{\beta k}{2\pi}}\, e^{-\beta k x^2/2}$

$\alpha_0(x) = 1 \qquad ; \quad \psi_0(x) = \sqrt{\pi(x)}$

(b) from 3/12 lecture notes:

$$C_A(t) = \sum_n \chi_n^A\, e^{\lambda_n |t|}$$

$$\chi_n^A = \langle A | n \rangle \langle n | A\pi \rangle = \int A(x)\, \varphi_n(x)\, dx \cdot \int \alpha_n(x)\, A(x)\, \pi(x)$$

$$= \int \psi_0(x)\, A(x)\, \psi_n(x) \cdot \int \psi_0(x)\, A(x)\, \psi_n(x) = (0|A|n)^2$$

bra-ket notation for ψ_n's

using $\varphi_n(x) = \sqrt{\pi(x)}\, \psi_n(x) = \psi_0(x)\, \psi_n(x)$

$$\alpha_n(x) = \frac{\psi_n(x)}{\sqrt{\pi(x)}} = \frac{\psi_n(x)}{\psi_0(x)}$$

$A = x$: $\quad (0|x|n) = \sqrt{\frac{\hbar}{2m\omega}}\, (0|a^\dagger + a|n) = \sqrt{\frac{\hbar}{2m\omega}}\, \delta_{n1} = \sqrt{\frac{1}{\beta k}}\, \delta_{n1}$

$$C_x(t) = \sum_n (0|x|n)^2\, e^{\lambda_n |t|} = \boxed{\frac{1}{\beta k}\, e^{-\beta D k |t|}}$$

agrees w/ O.U. results

c $\dfrac{\partial A}{\partial t} = \hat{\mathcal{L}}^{+} A \quad \rightarrow \quad A(t) = e^{\hat{\mathcal{L}}^{+} t} A(0)$

bra-ket notation: $|n\rangle \longleftrightarrow \alpha_n(x)$

$\langle n| \longleftrightarrow \varphi_n(x)$

$$e^{\hat{\mathcal{L}}^{+} t} |A\rangle = \sum_n e^{\lambda_n t} |n\rangle\langle n|A\rangle$$

$$= \sum_n e^{\lambda_n t} \alpha_n(x) \int dx\, \varphi_n(x) A(x)$$

$$\int dx\, \varphi_n(x) A(x) = \int dx\, \psi_0(x) A(x) \psi_n(x) = (0|A|n)$$

$A = x$: $(0|x|n) = \sqrt{\dfrac{1}{\beta k}}\, \delta_{n1}$ (see b)

$A = x^2$: $(0|x^2|n) = \dfrac{1}{\beta k} (0|(a^{+}+a)^2|n) = \dfrac{1}{\beta k}\left(\sqrt{2}\, \delta_{n2} + \delta_{n0}\right)$

$\alpha_0(x) = 1$, $\alpha_1(x) = \sqrt{\beta k}\, x$, $\alpha_2(x) = \sqrt{\tfrac{1}{2}}\,(\beta k x^2 - 1)$

(using results of a)

$$x(t) = \sum_n e^{\lambda_n t} \alpha_n(x) (0|x|n) = e^{-\beta D k t} \sqrt{\beta k}\, x \frac{1}{\sqrt{\beta k}} = \boxed{x e^{-\beta D k t}}$$

$$x^2(t) = \alpha_0(x)\, (0|x^2|0) + e^{-2\beta D k t} \alpha_2(x)\, (0|x^2|2)$$

$$= \frac{1}{\beta k} + e^{-2\beta D k t} \sqrt{\tfrac{1}{2}}\, (\beta k x^2 - 1) \frac{\sqrt{2}}{\beta k}$$

$$= \boxed{\frac{1}{\beta k}\left(1 - e^{-2\beta D k t}\right) + x^2 e^{-2\beta D k t}}$$

3 (a) $\mathcal{R} = U J V^{+}$ $\qquad UV^{+} = V^{+}U = I$

$$U = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$J = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right) \equiv \left(\begin{array}{c|c} 0 & \\ \hline & M \end{array}\right)$$

$$V^{+} = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

$$e^{\mathcal{R}t} = \sum_{k=0}^{\infty} \frac{\mathcal{R}^k t^k}{k!} = U\left(\sum_k \frac{t^k}{k!} J^k\right) V^{+}$$

$$= U \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & te^{-t} \\ 0 & 0 & e^{-t} \end{pmatrix} V^{+}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{6}e^{-t}\left(\begin{array}{c|c|c} 4-t & 2(t-1) & -(2+t) \\ -2 & 4 & -2 \\ -2+t & -2(t+1) & (4+t) \end{array}\right)$$

(b) $$\langle A_0 A_t \rangle = \sum_{mn} a_n \left(e^{\mathcal{R}t}\right)_{nm} a_m \pi_m$$

$$= (1 \;\; 1 \;\; -2) \cdot e^{\mathcal{R}t} \cdot \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} = \left(2 + \frac{t}{2}\right) e^{-t}$$

using $\pi = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

Since $\langle A \rangle = 0$, we get $\boxed{C_A(t) = \left(2 + \frac{t}{2}\right) e^{-t}}$